Appendix B: *Uncertainties*

A. Uncertainties in Measurements

1. Digital Readings

Digital displays look “exact”, but they are not, because they either truncate or round all digits they do not display. The error in what they display is that rounding, as much as half of the finest digit they do show. So the measurement’s *uncertainty* is ½ their finest digit shown.

What if the finest digit in its display is unstable, so it “wobbles” with a noticeable variation? If the displayed values range from (say) 220 mm to 236 mm , the uncertainty in any single value is not just ½ mm, but rather is at least half that range … ½ (236 – 220) mm = 8 mm . The uncertainty in the average value can be reduced by averaging many values (see B).

Digital clocks (including timers & stopwatches) truncate their displays. This shifts their range of error, to be centered at half of their finest displayed digit (which is their average error). Countdown timers always truncate UP, stopwatches and countup timers truncate DOWN. In most situations the truncation is ignored – if the truncation will be important, pay for a timer that displays an extra (finer) digit or two. The uncertainty is still ½ their finest digit.

What ~~if~~ when lab partners, trying to measure the same time interval, find different values on their stop-watches? Of course you average those values (unless one knows they goovved)! Treat the uncertainty as half the range of the values that were averaged.
With 3 (or more) timers, average all the values (that were not blunders!) … but measure the range for the ⅔ of values that are nearest that average (not the extreme outliers).

2. Analog Readings

For analog measurements you decide the digits to record by where a needle or other mark is. Some widely-spaced marks let you estimate halves, or thirds or quarters or fifths of them. If you’re “pretty sure” that it is ⅗ instead of ⅖ or ⅘, then you’re certain of the fifths – but not sure of the tenths. The uncertainty is half of the finest part that you are sure about.

Some markings are so fine (or poorly aligned) that it is not obvious which mark is indicated. If it is either this left one or that next right one, then your uncertainty is half that spacing. If it might be any of those 3 marks, you record the middle one as a best value, and your uncertainty spans a full division. It’s about being honest that you can’t tell for sure.

3. Reporting Uncertainties

Our uncertainties are half the finest digit, or half the range, because most situations are close to symmetric around the “best value” (average) – as likely to be larger as it is to be smaller.

So uncertainties are written as a range above and below that best value:
motion detector distance to cart ½ (220+236) = 228 ± 8mm

4. Statistical Uncertainties

Scientists expect that if all conditions are the same, they'll get the same measurement result. But with low-uncertainty measurements, this does not occur – all conditions are never exactly the same (Heraclitus). Whether the cylinder is not round, or the caliper has warmed during use, repeated measurements vary. We do multiple measurements because their average (mean) is more accurate (closer to "correct") than any one measure ... unless there is a consistent (systematic) bias as the lower throws illustrate.

Accurate means that the average is "good" – will be "close to" another sample's average. But each measurement try will deviate from their averages.

Precise means that these trials do not vary, "much". The *standard deviation* is the average of all these deviations, added in quadrature (*via* Pythagoras, they're independent). Deviations much bigger than the uncertainty suggest bad technique or systematic bias in sampling.

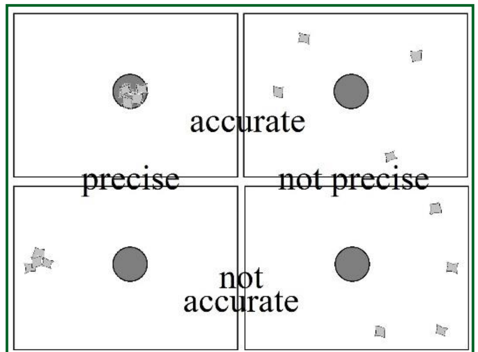

Figure B-1

Imprecise measurements need a lot more of them in the sample, to be confident in the average – if the 2 next throws are wide left, the two right boards switch their accuracy categories!

To use a measurement x in validating a relationship, its total uncertainty ux combines the reading uncertainty δx and the standard deviation σx, added in quadrature $ux = \sqrt{\delta x^2 + \sigma x^2}$ … if one contribution dominates, you can usually ignore the other. They are independent contributions, so their errors might partially cancel, not just add up.

B. Propagating Uncertainties

1. Addition & Averages

if $C = L1 + W1 + L2 + W2$, with independent uncertainties uL_1, uW_1, uL_2, uW_2,
then $\upsilon C = \sqrt{uL_1^2 + uL_2^2 + uW_1^2 + uW_2^2}$ … $\approx 2\, uL_1$, if the uncertainties are all equal.

So measuring the same thing 4 times and adding them: their sum has double one's uncertainty, but the average (after dividing by 4) has only ½ of an individual one's likely error.

If average $A = \frac{x_1 + x_2 + \cdots + x_N}{N}$, from N measures x_i with the same uncertainty ux,
then $uA = \frac{ux}{\sqrt{N}}$ … it took twice as long to measure 9 values as it took to measure 4,
but the uncertainty in the average uA only slid from ½ ux to ⅓ ux.

If identical items can be set adjacent to each other, best to measure all 4 at once back-to-back. Then the measurement uncertainty is divided by the number 4 itself (not its square root). The result here is only about the average item, not any of the individual items.

2. Conversions

Unit Conversions can usually be written in "slope + intercept" math form … $y = \mathrm{m}\,x + \mathrm{b}$:

if $C = \frac{100}{180}F - 17.\overline{7}$ … with these numbers exact but a known uncertainty uF ,

then $uC = \frac{100}{180}\,uF$ … the smaller Celsius value has a smaller number for its uncertainty, because the unit itself (thermometer mark spacing) is almost twice as big.

0 °F is the same as –17.8 °C … that vertical intercept obscures the important information! the formula **makes** much **more sense** with the horizontal intercept: $C = \frac{100}{180}(F - 32)$.

A physically meaningful measurement has the same fractional (%) uncertainty in either unit.

$$\frac{\delta L}{L} \approx \frac{0.002"}{2.75"} = \frac{0.05mm}{69.86mm} \approx 0.0007 = 0.07\% \ .$$

3. Multiplication & Division

Fractional uncertainties add in quadrature, weighted by the number of times they are a factor:

if $V = \frac{\pi}{4}D^2H^1$ … with known diameter uncertainty uD and height uncertainty uH ,

then $\dfrac{uV}{V} = \sqrt{\left(2\,\frac{uD}{D}\right)^2 + \left(1\,\frac{uH}{H}\right)^2}$ … since 1, 2, 4, and π have no uncertainty .

The diameter's uncertainty is doubled because it is used (multiplied) twice in the formula; an error in the 1st diameter value can never cancel the error in the 2nd (equal) value. (Best to treat it as an elliptical cross-section so you measure diameters twice. The caliper orientations (≈90° apart) make the values (& their errors!) independent from each other.

Division follows the same form … write it as a –1 exponent , its fractional uncertainty adds just like it was multiplied by, since the $(-1)^2$ becomes +1.

Write a square root operator as everything inside the root, taken to the ½ power …because that exponent is squared, they don't contribute very much to the eventual uncertainty.

It is important to "clean up" your formula, to properly estimate the uncertainty in the result.

$a_c = \dfrac{v^2}{r} = \dfrac{(2\pi r)^2}{t^2\ r} = \dfrac{4\pi^2\ r}{t^2}$ … the r uncertainty counts <u>once</u> – not 3 times !