Web Solutions for

How to Read and Do Proofs

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An Introduction to Mathematical Thought Processes

Third Edition

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1.4 a. Hypothesis: n is a positive integer.
Conclusion: The sum of the first n positive integers is $n(n+1)/2$.
b. Hypothesis: r is a real and satisfies $r^2 = 2$.
Conclusion: r is irrational.
c. Hypothesis: p and q are positive real numbers with $\sqrt{pq} \neq (p+q)/2$.
Conclusion: $p \neq q$.

1.10 (T = true, F = false)

A	B	C	$(A \Rightarrow B)$	$(A \Rightarrow B) \Rightarrow C$
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	T	F
F	F	T	T	T
F	F	F	T	F

1.11 a. According to Table 1.1, you should try to show that A is true and B is false because this is the only time that "A implies B" is false.
b. From part (a), to show that the statement, "If $x > 0$, then $\log_7(x) > 0$" is false, you need a value for x so that the hypothesis A is true and the conclusion B is false. Any value of x with $0 < x < 1$ does this.

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Web Solutions to Exercises

2.1 The forward process is a process that makes specific use of the information contained in the hypothesis A. The backward process is a process that tries to find a chain of statements leading to the fact that the conclusion B is true.

With the backward process, you start with the statement B that you are trying to conclude is true. By asking and answering key questions, you derive a sequence of new statements with the property that if the sequence of new statements is true, then B is true. The backward process continues until you obtain the statement A or until you can no longer ask and/or answer the key question.

With the forward process, you begin with the statement A that you assume is true. You then derive from A a sequence of new statements that are true as a result of A being true. Every new statement derived from A is directed toward linking up with the last statement obtained in the backward process. The last statement of the backward process acts as the guiding light in the forward process, just as the last statement in the forward process helps you choose the right key question and answer.

2.8 a. How can I show that an integer (namely, n^2) is even?
How can I show that the square of an integer (namely, n) is even?

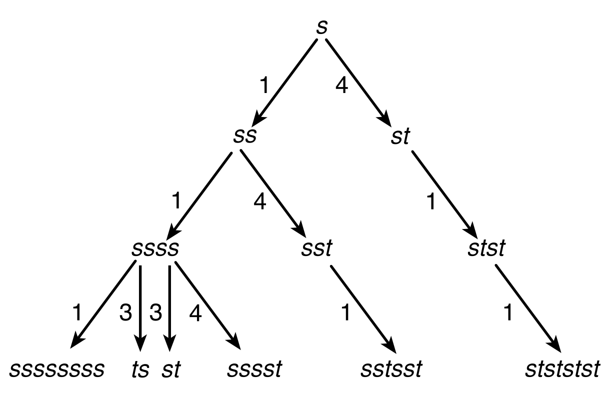
b. How can I show that an integer (namely, n) satisfies a given equation (namely, $2n^2 - 3n = -2$)?
How can I show that an integer (namely, n) is the root of a quadratic equation (namely, $2n^2 - 3n + 2$)?

2.11 a. Show that the product of the slopes of the two lines is -1.
Show that the two lines intersect at a 90 degree angle.
Show that the two lines are the legs of a right triangle.
b. Show that the elements of the two sets are identical.
Show that each set is a subset of the other.
Show that both sets are equal to a third set.

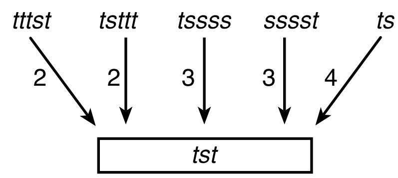
2.13 a. (1) How can I show that the solution to a quadratic equation is positive?
(2) Show that the quadratic formula gives a positive solution.
(3) Show that the solution $-b/2a$ is positive.
b. (1) How can I show that a triangle is equilateral?
(2) Show that the three sides have equal length (or show that the three angles are equal).
(3) Show that $\overline{RT} = \overline{ST} = \overline{SR}$ (or show that $\angle R = \angle S = \angle T$).

2.20 For sentence 1: The fact that $c^n = c^2c^{n-2}$ follows by algebra. The author then substitutes $c^2 = a^2 + b^2$, which is true from the Pythagorean theorem applied to the right triangle.
For sentence 2: For a right triangle, the hypotenuse c is longer than either of the two legs a and b so, $c > a$, $c > b$. Because $n > 2$, $c^{n-2} > a^{n-2}$ and $c^{n-2} > b^{n-2}$ and so, from sentence 1, $c^n = a^2c^{n-2} + b^2c^{n-2} > a^2(a^{n-2}) + b^2(b^{n-2})$.
For sentence 3: Algebra from sentence 2.

2.22 a. The number to the left of each line in the following figure indicates which rule is used.



b. The number to the left of each line in the following figure indicates which rule is used.



c.
$$\begin{array}{rll} A: & s & \text{given} \\ A1: & ss & \text{rule 1} \\ A2: & ssss & \text{rule 1} \\ B1: & sssst & \text{rule 4} \\ B: & tst & \text{rule 3} \end{array}$$

d.
$$\begin{array}{rll} A: & s & \text{given} \\ A1: & tst & \text{from part (c)} \\ A2: & tsttst & \text{rule 1} \\ A3: & tsst & \text{rule 2} \\ A4: & tssttsst & \text{rule 1} \\ B1: & tsssst & \text{rule 2} \\ B: & ttst & \text{rule 3} \end{array}$$

2.25 **Analysis of Proof.** A key question associated with the conclusion is, "How can I show that a triangle is equilateral?" One answer is to show that all three sides have equal length, specifically,

$$B1: \ \overline{RS} = \overline{ST} = \overline{RT}.$$

To see that $\overline{RS} = \overline{ST}$, work forward from the hypothesis to establish that

$B2:$ triangle RSU is congruent to triangle SUT.

Specifically, from the hypothesis, SU is a perpendicular bisector of RT, so

$$A1: \ \overline{RU} = \overline{UT}.$$

In addition,

$$A2: \ \angle RUS = \angle SUT = 90^o.$$
$$A3: \ \overline{SU} = \overline{SU}.$$

Thus the side-angle-side theorem states that the two triangles are congruent and so B2 has been established.

It remains (from $B1$) to show that

$$B3: \ \overline{RS} = \overline{RT}.$$

Working forward from the hypothesis you can obtain this because

$$A4: \ \overline{RS} = 2\overline{RU} = \overline{RU} + \overline{UT} = \overline{RT}.$$

Proof. To see that triangle RST is equilateral, it will be shown that $\overline{RS} = \overline{ST} = \overline{RT}$. To that end, the hypothesis that SU is a perpendicular bisector of RT ensures (by the side-angle-side theorem) that triangle RSU is congruent to triangle SUT. Hence, $\overline{RS} = \overline{ST}$. To see that $\overline{RS} = \overline{RT}$, by the hypothesis, one can conclude that $\overline{RS} = 2\overline{RU} = \overline{RU} + \overline{UT} = \overline{RT}$. □

3

Web Solutions to Exercises

3.3 (A is the hypothesis and $A1$ is obtained by working forward one step.)

a. $A:$ n is an odd integer.
$A1:$ $n = 2k + 1$, where k is an integer.

b. $A:$ s and t are rational numbers with $t \neq 0$.
$A1:$ $s = p/q$, where p and q are integers with $q \neq 0$. Also, $t = a/b$, where $a \neq 0$ and $b \neq 0$ are integers.

c. $A:$ Triangle RST is equilateral.
$A1:$ $\overline{RS} = \overline{ST} = \overline{RT}$, and $\angle R = \angle S = \angle T$.

d. $A:$ $\sin(X) = \cos(X)$.
$A1:$ $x/z = y/z$ (or $x = y$).

e. $A:$ a, b, c are integers for which $a|b$ and $b|c$.
$A1:$ $b = pa$ and $c = qb$, where p and q are two integers.

3.6 (T = true, F = false)

a. Truth Table for "*A OR B*."

A	B	"A OR B"
T	T	T
T	F	T
F	T	T
F	F	F

b. Truth Table for "A AND B."

A	B	"A AND B"
T	T	T
T	F	F
F	T	F
F	F	F

c. Truth Table for "A AND NOT B"

A	B	NOT B	"A AND NOT B"
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

d. Truth Table for "(NOT A) OR B."

A	NOT A	B	"(NOT A) OR B"	"A Implies B"
T	F	T	T	T
T	F	F	F	F
F	T	T	T	T
F	T	F	T	T

"A implies B" and "(NOT A) OR B" are equivalent. Both are false when A is true and B is false and true otherwise.

3.9 **Analysis of Proof.** The forward-backward method gives rise to the key question, "How can I show an integer (namely, n^2) is odd?" The definition for an odd integer is used to answer the key question, which means you have to show that

$B1:\ n^2 = 2k + 1$, for some integer k.

The forward process is now used to reach the desired conclusion.

Because n is an odd integer, by definition,

$A1:\ n = 2m + 1$, for some integer m.

Therefore,

$A2:\ n^2 = (2m + 1)^2$, so
$A3:\ n^2 = 4m^2 + 4m + 1$, so
$A4:\ n^2 = 2(2m^2 + 2m) + 1$.

Thus the desired value for k in $B1$ is $2m^2 + 2m$, and because m is an integer, $2m^2 + 2m$ is an integer. Hence it has been shown that n^2 can be expressed as 2 times some integer plus 1, which completes the proof.

Proof. Because n is an odd integer, there is an integer m for which $n = 2m+1$, and therefore,

$$\begin{aligned} n^2 &= (2m+1)^2 \\ &= 4m^2 + 4m + 1 \\ &= 2(2m^2 + 2m) + 1. \end{aligned}$$

Hence n^2 is an odd integer. □

3.13 To show that A is equivalent to B, one is led to the key question, "How can I show that two statements are equivalent?" By definition, it must be shown that

$B1$: "A implies B" and "B implies A".

The hypothesis states that "A implies B" so it remains to show that

$B2$: "B implies A".

This, however, follows from the hypothesis that "B implies C" and "C implies A" so, by the result in Exercise 3.12, "B implies A" and hence $B2$ is true.

Likewise, to show that A is equivalent to C, one is led to the key question, "How can I show that two statements are equivalent?" By definition, it must be shown that

$B1$: "A implies C" and "C implies A".

The hypothesis states that "C implies A" so it remains to show that

$B2$: "A implies C".

This, however, follows from the hypothesis that "A implies B" and "B implies C" so, by the result in Exercise 3.12, "A implies C" and hence $B2$ is true. The proof is now complete.

3.15 a. If the four statements in part (a) are true, then you can show that A is equivalent to any of the alternatives by using Exercise 3.13. For instance, to show that A is equivalent to D, you already know that "D implies A." By Exercise 3.13, because "A implies B," "B implies C," and "C implies D," you have that "A implies D."

b. The advantage of the approach in part (a) is that only four proofs are required ($A \Rightarrow B$, $B \Rightarrow C$, $C \Rightarrow D$, and $D \Rightarrow A$) as opposed to the six proofs ($A \Rightarrow B$, $B \Rightarrow A$, $A \Rightarrow C$, $C \Rightarrow A$, $A \Rightarrow D$, and $D \Rightarrow A$) required to show that A is equivalent to each of the three alternatives.

3.17 **Analysis of Proof.** The forward-backward method gives rise to the key question, "How can I show that a triangle is isosceles?" Using the definition of an isosceles triangle, you must show that two of its sides are equal which, in this case, means you must show that

$$B1: \ u = v.$$

Working forward from the hypothesis, you know that

$$A1: \ \sin(U) = \sqrt{u/2v}.$$

By the definition of sine, $\sin(U) = u/w$, so

$$A2: \ \sqrt{u/2v} = u/w,$$

and by algebraic manipulations, you obtain

$$A3: \ w^2 = 2uv.$$

Furthermore, by the Pythagorean theorem,

$$A4: \ u^2 + v^2 = w^2.$$

Substituting for w^2 from $A3$ in $A4$, you have that

$$A5: \ u^2 + v^2 = 2uv, \text{ or, } u^2 - 2uv + v^2 = 0.$$

On factoring $A5$ and then taking the square root of both sides of the equality, it follows that

$$A6: \ u - v = 0$$

and so $u = v$, completing the proof.

Proof. Because $\sin(U) = \sqrt{u/2v}$ and also $\sin(U) = u/w$, $\sqrt{u/2v} = u/w$, or, $w^2 = 2uv$. Now from the Pythagorean theorem, $w^2 = u^2+v^2$. On substituting $2uv$ for w^2 and then performing algebraic manipulations, one has $u = v$. □

3.19 **Analysis of Proof.** To verify the hypothesis of Proposition 3 for the current triangle UVW, it is necessary to match up the notation. Specifically, $r = u$, $s = v$, and $t = w$. Thus it must be shown that

$$B1: \ w = \sqrt{2uv}.$$

But, as in the proof in Exercise 3.18,

$$\begin{aligned} A1&: \ \sqrt{u/2v} = u/w, \text{ or,} \\ A2&: \ u/2v = u^2/w^2, \text{ or,} \\ A3&: \ w^2 = 2uv. \end{aligned}$$

On taking the positive square root of both sides of the equality in $A3$, one obtains precisely $B1$, thus completing the proof. (Observe also that triangle UVW is a right triangle.)

Proof. By the hypothesis, $\sin(U) = \sqrt{u/2v}$ and from the definition of sine, $\sin(U) = u/w$, thus one has $u/2v = u/w$. By algebraic manipulations one obtains $w = \sqrt{uv}$. Hence the hypothesis of Proposition 3 holds for the current right triangle UVW, and consequently the triangle is isosceles. □

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Web Solutions to Exercises

4.3 a. A triangle XYZ is isosceles if $\exists$ two sides $\ni$ their lengths are equal.
b. For a polynomial of degree n, say $p(x)$, $\exists$ exactly n complex numbers, $r_1, \ldots, r_n \ni p(r_1) = \cdots = p(r_n) = 0$.

4.7 **Analysis of Proof.** The appearance of the key words "there is" in the conclusion suggests using the construction method to find a real number x such that $x^2 - 5x/2 + 3/2 = 0$. Factoring this quadratic means you want to find a real number x such that $(x - 3/2)(x - 1) = 0$. So the desired real number is either $x = 1$ or $x = 3/2$ which, when substituted in $x^2 - 5x/2 + 3/2$ is 0. The real number is not unique as either $x = 1$ or $x = 3/2$ works.

Proof. Factoring $x^2 - 5x/2 + 3/2 = 0$ yields $(x - 3/2)(x - 1) = 0$, so $x = 1$ or $x = 3/2$. Thus, there exists a real number, namely, $x = 1$ or $x = 3/2$, such that $x^2 - 5x/2 + 3/2 = 0$. The real number is not unique. □

4.11 **Analysis of Proof.** The forward-backward method gives rise to the key question, "How can I show that a real number (namely, s/t) is rational?"" By the definition, one answer is to show that

$B1$: there are integers p and q with $q \neq 0$ such that $s/t = p/q$.

The appearance of the quantifier "there are" suggests turning to the forward process to construct the desired p and q.

From the hypothesis that s and t are rational numbers, and by the definition,

$A1:$ there are integers a, b, c, and d with $b \neq 0$ and $d \neq 0$ such that $s = a/b$ and $t = c/d$.

Because $t \neq 0$, $c \neq 0$, and thus $bc \neq 0$. Hence,

$A2:$ $s/t = (a/b)/(c/d) = (ad)/(bc)$.

So the desired integers p and q are $p = ad$ and $q = bc$. Observe that because $b \neq 0$ and $c \neq 0$, $q \neq 0$; also, $s/t = p/q$, thus completing the proof.

Proof. Because s and t are rational, there are integers a, b, c, and d with $b \neq 0$ and $d \neq 0$ such that $s = a/b$ and $t = c/d$.

Because $t \neq 0$, $c \neq 0$. Constructing $p = ad$ and $q = bc$, and noting that $q \neq 0$, one has $s/t = (a/b)/(c/d) = (ad)/(bc) = p/q$, and hence s/t is rational, thus completing the proof. □

4.14 The proof is not correct. The mistake occurs because the author uses the same symbol x for the element that is in $R \cap S$ and in $S \cap T$ where, in fact, the element in $R \cap S$ need not be the same as the element in $S \cap T$.

5

Web Solutions to Exercises

5.7 a. $\exists$ a mountain $\ni$ $\forall$ other mountains, this one is taller than the others.
b. $\forall$ angle t, $\sin(2t) = 2\sin(t)\cos(t)$.
c. $\forall$ nonnegative real numbers p and q, $\sqrt{pq} \geq (p+q)/2$.
d. $\forall$ real numbers x and y with $x < y$, $\exists$ a rational number $r \ni\ x < r < y$.

5.11 **Analysis of Proof.** The appearance of the quantifier "for every" in the conclusion suggests using the choose method, whereby one chooses

$A1:$ an element $t \in T$,

for which it must be shown that

$B1:$ t is an upper bound for the set S.

A key question associated with $B1$ is, "How can I show that a real number (t) is an upper bound for a set (S)?" By definition, one must show that

$B2:$ for every element $x \in S$, $x \leq t$.

The appearance of the quantifier "for every" in $B2$ suggests using the choose method, whereby one chooses

$A2:$ an element $x \in S$,

for which it must be shown that

$B3:$ $x \leq t$.

To do so, work forward from $A2$ and the definition of the set S in the hypothesis to obtain

$A3:\ x(x-3) \leq 0.$

From $A3$, either $x \geq 0$ and $x - 3 \leq 0$, or, $x \leq 0$ and $x - 3 \geq 0$. But the latter cannot happen, so

$A4:\ x \geq 0$ and $x - 3 \leq 0$.

From $A4$,

$A5:\ x \leq 3.$

But, from $A1$ and the definition of the set T in the hypothesis

$A6:\ t \geq 3.$

Combining $A5$ and $A6$ yields $B3$, thus completing the proof.

5.17 **Analysis of Proof.** The forward-backward method gives rise to the key question, "How can I show that a set (namely, C) is convex?" One answer is by the definition, whereby it must be shown that

$B1:$ for all elements x and y in C, and for all real numbers t with $0 \leq t \leq 1, tx + (1-t)y \in C$.

The appearance of the quantifiers "for all" in $B1$ suggests using the choose method to choose

$A1:$ elements x and y in C, and a real number t with $0 \leq t \leq 1$,

for which it must be shown that

$B2:\ tx + (1-t)y \in C$, that is, $a[tx + (1-t)y] \leq b$.

Turning to the forward process, because x and y are in C (see A1),

$A2:\ ax \leq b$ and $ay \leq b$.

Multiplying both sides of the two inequalities in $A2$, respectively, by the nonnegative numbers t and $1-t$ (see $A1$) and adding the inequalities yields:

$A3:\ tax + (1-t)ay \leq tb + (1-t)b.$

Performing algebraic manipulations on $A3$ yields $B2$, thus completing the proof.

Proof. Let t be a real number with $0 \leq t \leq 1$, and let x and y be in C. Then $ax \leq b$ and $ay \leq b$. Multiplying both sides of these inequalities by $t \geq 0$ and $1-t \geq 0$, respectively, and adding yields $a[tx + (1-t)y] \leq b$. Hence, $tx + (1-t)y \in C$. So, for every real number t with $0 \leq t \leq 1$, and for all elements x and y in C, $tx + (1 - t)y \in C$. Therefore, C is a convex set and the proof is complete. $\square$

6

Web Solutions to Exercises

6.1 The reason you need to show that Y has the certain property is because you only know that the something happens for objects with the certain property. You do not know that the something happens for objects that do not satisfy the certain property. Therefore, if you want to use specialization to claim that the something happens for this particular object Y, you must be sure that Y has the certain property.

6.11 **Analysis of Proof.** The appearance of the quantifier "for all" in the conclusion indicates that you should use the choose method to choose

$A1$: a real number $s' \geq 0$,

for which it must be shown that

$B1$: the function $s'f$ is convex.

An associated key question is, "How can I show that a function (namely, $s'f$) is convex?" Using the definition in Exercise 5.1(e), one answer is to show that

$B2$: for all real numbers x and y, and for all t with $0 \leq t \leq 1$,
$s'f(tx + (1-t)y) \leq ts'f(x) + (1-t)s'f(y)$.

The appearance of the quantifier "for all" in the backward process suggests using the choose method to choose

$A2$: real numbers x' and y', and $0 \leq t' \leq 1$, for which it must be shown that

$$B3: \; s'f(t'x' + (1-t')y') \le t's'f(x') + (1-t')s'f(y').$$

The desired result is obtained by working forward from the hypothesis that f is a convex function. By the definition in Exercise 5.1(e), you have that

$A3:$ for all real numbers x and y, and for all t with $0 \le t \le 1$,
$f(tx + (1-t)y) \le tf(x) + (1-t)f(y)$.

Specializing the statement in $A3$ to $x = x'$, $y = y'$, and $t = t'$ (noting that $0 \le t' \le 1$) yields

$$A4: \; f(t'x' + (1-t')y') \le t'f(x') + (1-t')f(y').$$

The desired statement $B3$ is obtained by multiplying both sides of the inequality in $A4$ by the nonnegative number s', thus completing the proof.

Proof. Let $s' \ge 0$. To show that $s'f$ is convex, let x' and y' be real numbers, and let t' with $0 \le t' \le 1$. It will be shown that $s'f(t'x' + (1-t')y') \le t's'f(x') + (1-t')s'f(y')$.

Because f is a convex function by hypothesis, it follows from the definition that $f(t'x'+(1-t')y') \le t'f(x')+(1-t')f(y')$. The desired result is obtained by multiplying both sides of this inequality by the nonnegative number s'. □

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Web Solutions to Exercises

7.9 **Analysis of Proof.** The forward-backward method gives rise to the key question, "How can I show that a function (namely, f) is bounded above?" One answer is by the definition, whereby one must show that

$B1$: there is a real number y such that for every real number x, $-x^2 + 2x \leq y$.

The appearance of the quantifier "there is" in $B1$ suggests using the construction method to produce the desired value for y. Trial and error might lead you to construct $y = 1$ (any value of $y \geq 1$ will also work). Now it must be shown that this value of $y = 1$ is correct, that is:

$B2$: for every real number x, $-x^2 + 2x \leq 1$.

The appearance of the quantifier "for all" in $B2$ suggests using the choose method to choose

$A1$: a real number x,

for which it must be shown that

$B3$: $-x^2 + 2x \leq 1$, that is, $x^2 - 2x + 1 \geq 0$.

But because $x^2 - 2x + 1 = (x - 1)^2$, this number is always ≥ 0. Thus $B3$ is true, completing the proof.

Proof. To see that the function $f(x) = -x^2 + 2x$ is bounded above, it will be shown that for all real numbers x, $-x^2 + 2x \leq 1$. To that end, let x be any real number. Then $x^2 - 2x + 1 = (x-1)^2 \geq 0$, thus completing the proof. □

7.11 **Analysis of Proof.** The first key words in the conclusion from the left are "for all," so the choose method is used to choose

$$A1: \text{ a real number } \epsilon > 0,$$

for which it must be shown that

$$B1: \text{ there is an element } x \in S \text{ such that } x > 1 - \epsilon.$$

Recognizing the key words "there is" in $B1$, the construction method is used to produce the desired element in S. To that end, from the hint, you can write S as follows:

$$S = \{\text{real numbers } x : \text{there is an integer } n \geq 2 \text{ with } x = 1 - 1/n\}.$$

Thus, you can construct $x = 1 - 1/n$, for an appropriate choice of the integer $n \geq 2$. To find the value for n, from $B1$, you want

$$x = 1 - 1/n > 1 - \epsilon, \quad \text{that is,} \quad 1/n < \epsilon, \quad \text{that is,} \quad n > 1/\epsilon.$$

In summary, noting that $\epsilon > 0$ from $A1$, if you let $n > 2$ be any integer $> 1/\epsilon$, then $x = 1 - 1/n \in S$ satisfies the desired property in $B1$, namely, that $x = 1 - 1/n > 1 - \epsilon$. The proof is now complete.

Proof. Let $\epsilon > 0$. To see that there is an element $x \in S$ such that $x > 1 - \epsilon$, let $n > 2$ be an integer for which $n > 1/\epsilon$. It then follows from the defining property that $x = 1 - 1/n \in S$ and by the choice of n that

$$x = 1 - 1/n > 1 - 1/\epsilon.$$

It has thus been shown that for every real number $\epsilon > 0$, there is an element $x \in S$ such that $x > 1 - \epsilon$, thus completing the proof. □

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Web Solutions to Exercises

8.11 **Analysis of Proof.** By contradiction, assume A and $NOT\ B$, that is:

$$A: \quad n-1,\ n, \text{ and } n+1 \text{ are consecutive positive integers.}$$
$$A1\ (NOT\ B): \quad (n+1)^3 = n^3 + (n-1)^3.$$

A contradiction is reached by showing that

$$B1: \ n^2(n-6) = 2 \text{ and } n^2(n-6) \geq 49.$$

To that end, rewriting $A1$ by algebra, you have:

$$A2: \quad n^3 + 3n^2 + 3n + 1 = n^3 + n^3 - 3n^2 + 3n - 1, \text{ or}$$
$$A3: \quad n^3 - 6n^2 = 2, \text{ or}$$
$$A4: \quad n^2(n-6) = 2.$$

From $A4$, because $n^2 > 0$, it must be that

$$A5: \ n - 6 > 0, \text{ that is, } n \geq 7.$$

But when $n \geq 7$, $n - 6 \geq 1$, and $n^2 \geq 49$, so

$$A6: \ n^2(n-6) \geq n^2 \geq 49.$$

Now $A6$ contradicts $A4$ because $A6$ states that $n^2(n-6) \geq 49$ while $A4$ states that $n^2(n-6) = 2$. This contradiction completes the proof.

Proof. Assume, to the contrary, that the three consecutive integers $n-1$, n, and $n+1$ satisfy

$$(n+1)^3 = n^3 + (n-1)^3.$$

Expanding these expressions and rewriting yields

$$n^2(n-6) = 2.$$

Because $n^2 > 0$, $n-6 > 0$, that is, $n \geq 7$. But then $n^2(n-6) \geq n^2 \geq 49$. This contradicts the fact that $n^2(n-6) = 2$, thus completing the proof. □

8.15 **Analysis of Proof.** By the contradiction method, it can be assumed that

$A1:$ the number of primes is finite.

From $A1$, there will be a prime number that is larger than all the other prime numbers. So,

$A2:$ let n be the largest prime number.

Consider the number $n!+1$, and let

$A3:$ p be any prime number that divides $n!+1$.

A contradiction is reached by showing that

$B1:$ $p \leq n$ and $p > n$.

Because n is the largest prime and p is prime, $p \leq n$. It remains to show that $p > n$. To do so, use the fact that p divides $n!+1$ to show that

$B2:$ $p \neq 2$, $p \neq 3$, $\ldots$, $p \neq n$.

To see that $B2$ is true, observe that when $n!+1$ is divided by 2, there is a remainder of 1 because,

$$n!+1 = [n(n-1)...(2)(1)] + 1.$$

Similarly, when $n!+1$ is divided by r, where $1 < r \leq n$, there is a remainder of 1. Indeed, one has that

$$\frac{n!+1}{r} = \frac{n!}{r} + \frac{1}{r}.$$

Hence $n!+1$ has no prime factor between 1 and n. Therefore the prime factor p is greater than n, which contradicts the assumption that n is the largest prime number, thus completing the proof.

Proof. Assume, to the contrary, that there are a finite number of primes. Let n be the largest prime. Let p be any prime divisor of $n!+1$. Because n is the largest prime, p must be less than or equal to n. But $n!+1$ cannot be divided by any number between 1 and n. Hence $p > n$, which is a contradiction. □

8.19 **Analysis of Proof.** Assume, to the contrary, that

$A1$ ($NOT\ B$): the polynomial x^4+2x^2+2x+2 can be expressed as the product of the two polynomials x^2+ax+b and x^2+cx+d in which a, b, c, and d are integers.

Working forward by multiplying the two polynomials, you have that

$$A2: \ x^4+2x^2+2x+2=(x^2+ax+b)(x^2+cx+d)= \\ x^4+(a+c)x^3+(b+ac+d)x^2+(bc+ad)x+bd.$$

Equating coefficients of like powers of x on both sides, it follows that

$A3:$ $a+c=0.$
$A4:$ $b+ac+d=2.$
$A5:$ $bc+ad=2.$
$A6:$ $bd=2.$

From $A6$, b is odd (±1) and d is even (±2) or vice versa. Suppose, first, that

$A7:$ **Case 1:** b is odd and d is even.

(Subsequently, the case when b is even and d is odd is considered.) It now follows that, because the right side of $A5$ is even, the left side is also. Because d is even, so is ad. It must therefore be the case that

$A8:$ bc is even.

However, from $A7$, b is odd, so it must be that

$A9:$ c is even.

But then,

$A10:$ the left side of $A4$ = odd + even + even = odd.

This is because b is odd (see $A7$), ac is even (from $A9$), and d is even (see $A7$). However, $A10$ is a contradiction because the right side of $A4$, namely, 2, is even. This establishes a contradiction for Case 1 in $A7$. A similar contradiction is reached in Case 2, when b is even and d is odd, thus completing the proof.

9

Web Solutions to Exercises

9.9 **Analysis of Proof.** With the contrapositive method, you assume that

> $A1$ (*NOT* B): there is an integer solution, say m, to the equation $n^2 + n - c = 0$.

It must be shown that

> $B1$ (*NOT* A): c is not odd, that is, c is even.

But from $A1$, you have that

> $A2$: $c = m + m^2$.

Observe that $m + m^2 = m(m+1)$ is the product of two consecutive integers and is therefore even, thus establishing $B1$ and completing the proof.

Proof. Assume that there is an integer solution, say m, to the equation $n^2 + n - c = 0$. It will be shown that c is even. But $c = m + m^2 = m(m+1)$ is even because the product of two consecutive integers is even. □

9.11 **Analysis of Proof.** By the contrapositive method, you can assume that

> $A1$ (*NOT* B): the quadrilateral $RSTU$ is not a rectangle.

It must be shown that

> $B1$ (*NOT* A): there is an obtuse angle.

The appearance of the quantifier "there is" in $B1$ suggests turning to the forward process to produce the obtuse angle.

Working forward from $A1$, you can conclude that

> $A2$: at least one angle of the quadrilateral is not 90 degrees, say, angle R.

If angle R has more than 90 degrees, then it is the desired angle and the proof is complete. Otherwise,

> $A3$: angle R has less than 90 degrees.

This in turn means that

> $A4$: the remaining angles of the quadrilateral must add up to more than 270 degrees,

because the sum of all the angles in $RSTU$ is 360 degrees. Among these three angles that add up to more than 270 degrees, one of them must be greater than 90 degrees, and that is the desired obtuse angle. The proof is now complete.

Proof. Assume that the quadrilateral $RSTU$ is not a rectangle, and hence, one of its angles, say, R, is not 90 degrees. An obtuse angle will be found. If angle R has more than 90 degrees, then it is the desired obtuse angle. Otherwise the remaining three angles add up to more than 270 degrees. Thus one of the remaining three angles is obtuse. □

10

Web Solutions to Exercises

10.7 **Analysis of Proof.** By the contrapositive method, you can assume

$A1$ (NOT B): $x < 0$.

It must be shown that

$B1$ (NOT A): there is a real number $\epsilon > 0$ such that $x < -\epsilon$.

Recognizing the key words "there is" in $B1$, the construction method is used to produce the desired $\epsilon > 0$. Turning to the forward process to do so, from $A1$, because $x < 0$, construct ϵ as any value with $0 < \epsilon < -x$. (Note that this construction is possible because $-x > 0$.) By design, $\epsilon > 0$ and, because $\epsilon < -x$, it follows that $x < -\epsilon$. Thus ϵ has all the needed properties in $B1$, and the proof is complete.

Proof. Assume, to the contrary, that $x < 0$. It will be shown that there is a real number $\epsilon > 0$ such that $x < -\epsilon$. To that end, construct ϵ as any value with $0 < \epsilon < -x$ (noting that this is possible because $-x > 0$). Clearly $\epsilon > 0$ and because $\epsilon < -x$, $x < -\epsilon$, thus completing the proof. □

10.9 **Analysis of Proof.** When using the contradiction method, you can assume the hypothesis that

$A:$ $x \geq 0$, $y \geq 0$, $x + y = 0$,

and also that

$A1$ (NOT B): either $x \neq 0$ or $y \neq 0$.

From $A1$, suppose first that

$$A2: \ x \neq 0.$$

Because $x \geq 0$ from A, it must be that

$$A3: \ x > 0.$$

A contradiction to the fact that $y \geq 0$ is reached by showing that

$$B1: \ y < 0.$$

Specifically, because $x + y = 0$ from A,

$$A4: \ y = -x.$$

Because $-x < 0$ from $A3$, a contradiction has been reached. A similar argument applies for the case where $y \neq 0$ (see $A1$).

Proof. Assume that $x \geq 0$, $y \geq 0$, $x + y = 0$, and that either $x \neq 0$ or $y \neq 0$. If $x \neq 0$, then $x > 0$ and $y = -x < 0$, but this contradicts the fact that $y \geq 0$. Similarly, if $y \neq 0$, then $y > 0$, and $x = -y < 0$, but this contradicts the fact that $x \geq 0$. $\square$

11

Web Solutions to Exercises

11.3 **Analysis of Proof.** According to the indirect uniqueness method, one must first construct a real number x for which $mx + b = 0$. But because the hypothesis states the $m \neq 0$, you can construct

$$A1: \; x = -b/m.$$

This value is correct because

$$A2: \; mx + b = m(-b/m) + b = -b + b = 0.$$

To establish the uniqueness by the indirect uniqueness method, suppose that

$$A3: \; y \text{ is a real number with } y \neq x \text{ such that } my + b = 0.$$

A contradiction to the hypothesis that $m \neq 0$ is reached by showing that

$$B1: \; m = 0.$$

Specifically, from $A2$ and $A3$,

$$A4: \; mx + b = my + b.$$

Subtracting the right side of the equality in $A4$ from the left side and rewriting yields

$$A5: \; m(x - y) = 0.$$

On dividing both sides of the equality in $A5$ by the nonzero number $x-y$ (see $A3$), it follows that $m = 0$. This contradiction establishes the uniqueness.

Proof. To construct the number x for which $mx + b = 0$, let $x = -b/m$ (because $m \neq 0$). Then $mx + b = m(-b/m) + b = 0$.

Now suppose that $y \neq x$ and also satisfies $my+b = 0$. Then $mx+b = my+b$, and so $m(x-y) = 0$. But because $x - y \neq 0$, it must be that $m = 0$, and this contradicts the hypothesis that $m \neq 0$. □

11.8 a. The time to use induction instead of the choose method to show that, "for every integer $n \geq n_0$, $P(n)$ is true" is when you can relate $P(n+1)$ to $P(n)$, for then you can use the induction hypothesis that $P(n)$ is true, and this should help you establish that $P(n+1)$ is true. If you were to use the choose method, you would choose

$A1:$ an integer $n \geq n_0$,

for which it must be shown that

$B1:$ $P(n)$ is true.

With the choose method, you cannot use the assumption that $P(n-1)$ is true to do so.

b. It is not possible to use induction when the object is not an integer because showing that $P(n)$ implies $P(n+1)$ may "skip over" many values of the object. As a result, the statement will not have been proved for such values.

11.12 **Proof.** First it is shown that $P(n)$ is true for $n = 5$. But $2^5 = 32$ and $5^2 = 25$, so $2^5 > 5^2$. Hence $P(n)$ is true for $n = 5$. Assuming that $P(n)$ is true, you must then prove that $P(n+1)$ is true. So assume

$P(n):\ 2^n > n^2.$

It must be shown that

$P(n+1):\ 2^{n+1} > (n+1)^2.$

Starting with the left side of $P(n+1)$, and using the fact that $P(n)$ is true, you have:

$$2^{n+1} = 2(2^n) > 2(n^2).$$

To obtain $P(n+1)$, it must still be shown that for $n > 5$, $2n^2 > (n+1)^2 = n^2+2n+1$, or, by subtracting n^2+2n-1 from both sides and factoring, that $(n-1)^2 > 2$. This last statement is true because, for $n > 5$, $(n-1)^2 \geq 4^2 = 16 > 2$. □

11.18 **Proof.** For $n = 1$ the statement becomes:

$P(1):\ [\cos(x) + i\sin(x)]^1 = \cos(1x) + i\sin(1x).$

Now $P(1)$ is true because $\cos(x) + i\sin(x) = \cos(x) + i\sin(x)$.

Now assume the statement is true for n, that is:

$$P(n): \ [\cos(x) + i\sin(x)]^n = \cos(nx) + i\sin(nx).$$

It must be shown that $P(n+1)$ is true, that is:

$$P(n+1): \ [\cos(x) + i\sin(x)]^{n+1} = \cos((n+1)x) + i\sin((n+1)x).$$

Starting with the left side of $P(n+1)$,

$$[\cos(x) + i\sin(x)]^{n+1} = [\cos(x) + i\sin(x)]^n[\cos(x) + i\sin(x)].$$

Using the fact that $P(n)$ is true and the facts that

$$\begin{aligned} \cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b), \\ \sin(a+b) &= \sin(a)\cos(b) + \cos(a)\sin(b), \end{aligned}$$

you have:

$$\begin{aligned} [\cos(x) + i\sin(x)]^{n+1} &= [\cos(nx) + i\sin(nx)][\cos(x) + i\sin(x)] \\ &= [\cos(nx)\cos(x) - \sin(nx)\sin(x)] + \\ &\quad\ i[\sin(nx)\cos(x) + \cos(nx)\sin(x)] \\ &= \cos((n+1)x) + i\sin((n+1)x). \end{aligned}$$

This establishes that $P(n+1)$ is true, thus completing the proof. □

11.23 The author relates $P(n+1)$ to $P(n)$ by using the product rule of differentiation to express

$$[x(x^n)]' = (x)'(x^n) + x(x^n)'.$$

The author then uses the induction hypothesis to replace $(x^n)'$ with nx^{n-1}.

11.26 The proof is incorrect because when $n = 1$ and $r = 1$, the right side of $P(1)$ is undefined since you cannot divide by zero. A similar problem arises throughout the rest of the proof.

12

Web Solutions to Exercises

12.5 The author uses a proof by cases when the following statement containing the key words "either/or" is encountered in the forward process:

$A1:$ either the factor b is odd or the factor d is odd.

Accordingly, the author considers the following two cases:

Case 1: The factor b is odd. The author then works forward from this information to establish the contradiction that the left side of (2) is odd and yet is equal to the even number 2.

Case 2: The factor d is odd. The author claims, without providing details, that this case also leads to a contradiction.

12.8 **Analysis of Proof.** According to this either/or method, you can assume that

$$\begin{aligned} A:&\quad x^3 + 3x^2 - 9x - 27 \geq 0, \text{ and} \\ A1\ (NOT\ D):&\quad x < 3. \end{aligned}$$

It must be shown that

$B1\ (C)$: $x \leq -3$, or $x + 3 \leq 0$.

By factoring A, it follows that

$A2:$ $x^3 + 3x^2 - 9x - 27 = (x-3)(x+3)^2 \geq 0$.

From $A1$, because $x < 3$, $x - 3 < 0$. Thus, dividing both sides of $A2$ by $x - 3$ yields

$A3: \ (x+3)^2 \leq 0.$

Because $(x+3)^2$ is also ≥ 0, from $A3$, it must be that

$A4: \ (x+3)^2 = 0$, so, $x + 3 = 0$, or, $x = -3$.

Thus $B1$ is true, completing the proof.

Proof. Assume that $x^3 + 3x^2 - 9x - 27 \geq 0$ and $x < 3$. Then it follows that

$$x^3 + 3x^2 - 9x - 27 = (x-3)(x+3)^2 \geq 0.$$

Because $x < 3$, $(x+3)^2$ must be 0, so $x + 3 = 0$, or equivalently, $x = -3$. □

12.9 **Analysis of Proof.** Observe that the conclusion can be written as:

$B:$ either $a = b$ or $a = -b$.

The key words "either/or" in B now suggest proceeding with a proof by elimination. Accordingly, you can assume the hypothesis and

$A1: \ a \neq b.$

It must be shown that

$B1: \ a = -b.$

Working forward from the hypotheses that $a|b$ and $b|a$, by the definition it follows that:

$A2:$ there is an integer k such that $b = ka$,
$A3:$ there is an integer m such that $a = mb$.

Substituting $a = mb$ in the equality in $A2$ yields:

$A4: \ b = kmb.$

If $b = 0$ then, from $A3$, $a = 0$ and so $B1$ is clearly true and the proof is complete. Thus, you can assume that $b \neq 0$. Therefore, on dividing both sides of the equality in $A4$ by b you obtain:

$A5: \ km = 1.$

From $A5$ and the fact that k and m are integers (see $A2$ and $A3$), it must be that

$A6: \ (k = 1$ and $m = 1)$ or $(k = -1$ and $m = -1)$.

Recognizing the key words "either/or" in $A6$, a proof by cases is used.

Case 1: $k = 1$ and $m = 1$. In this case, $A2$ leads to $a = b$, which cannot happen according to $A1$.

Case 2: $k = -1$ and $m = -1$. In this case, from $A2$, it follows that $a = -b$, which is precisely $B1$, thus completing the proof.

Proof. To see that $a = \pm b$, assume that $a|b$, $b|a$, and $a \neq b$. It will be shown that $a = -b$. By definition, it follows that there are integers k and m such that $b = ka$ and $a = mb$. Consequently, $b = kmb$. If $b = 0$, then $a = mb = 0$ and so $a = -b$. Thus, assume that $b \neq 0$. It then follows that $km = 1$. Because k and m are integers, it must be that $k = m = 1$ or $k = m = -1$. However, because $a \neq b$, it must be that $k = m = -1$. From this it follows that $a = mb = -b$, and so the proof is complete. □

12.16 **Analysis of Proof.** The max/min method is used to convert the conclusion to the equivalent statement

$B1:$ for all $s \in S$, $s \geq t^*$.

The appearance of the quantifier "for all" in the backward process suggests using the choose method to choose

$A1:$ an element $s' \in S$,

for which it must be shown that

$B2:$ $s' \geq t^*$.

The desired conclusion is obtained by working forward. Specifically, because S is a subset of T, it follows by definition that

$A2:$ for all elements $s \in S$, $s \in T$.

Specializing $A2$ to $s = s'$, which is in S (see $A1$), it follows that

$A3:$ $s' \in T$.

Also, the hypothesis states that

$A4:$ for all $t \in T$, $t \geq t^*$.

Specializing $A4$ with $t = s'$, which is in T (see A3), it follows that

$A5:$ $s' \geq t^*$,

which is precisely $B2$ and so the proof is complete.

Proof. To reach the conclusion, let $s' \in S$. It will be shown that $s' \geq t^*$. By the hypothesis that $S \subseteq T$, it follows that $s' \in T$. But then the hypothesis that for all $t \in T$, $t \geq t^*$ ensures that $s' \geq t^*$. □

13

Web Solutions to Exercises

13.3 a. Using the induction method one would first have to show that $4! > 4^2$. Then one would assume that $n! > n^2$ and $n \geq 4$, and try to show that $(n+1)! > (n+1)^2$.

b. Using the choose method, one would choose an integer n' for which $n' \geq 4$. One would then try to show that $(n')! > (n')^2$.

c. Converting the statement to the form "if ... then ..." one obtains "if n is an integer ≥ 4, then $n! > n^2$." One would therefore assume that n is an integer ≥ 4 and try to show that $n! > n^2$.

d. Using the contradiction method, one would assume that there is an integer $n \geq 4$ such that $n! \leq n^2$, and try to reach a contradiction.