Filter quantifiers

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This note summarizes the *filter quantifiers*, which provide an extremely useful notational device in several areas of mathematics. The filter quantifier $(\forall_{\mathcal{F}}\, x)P(x)$ says that the set of x for which $P(x)$ holds is in the filter $\mathcal{F}$, while the dual quantifier $(\exists_{\mathcal{F}}\, x)P(x)$ says that the set of x for which $P(x)$ holds is stationary. These quantifiers generalize both the usual quantifiers $\forall$ and $\exists$ and the infinitary quantifiers $\forall^\infty$ and $\exists^\infty$.

1 Filters and largeness

Let W be a fixed infinite set with power set $\mathcal{P}(W)$.

Definition 1.1. A *filter* on W is a collection $\mathcal{F}$ of subsets of W such that:

1. *Nontriviality:* $\emptyset \notin \mathcal{F}$ and $W \in \mathcal{F}$.
2. *Consistency:* If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$.
3. *Monotonicity:* If $A \in \mathcal{F}$, $B \in \mathcal{P}(W)$, and $A \subseteq B$ then $B \in \mathcal{F}$.

The dual notion is an *ideal*, which contains $\emptyset$ but not W, and which is closed under taking unions and subsets instead of taking intersections and supersets.

The terms “filter” and “ideal” are also used in the context of partially ordered sets, and the definition above might initially appear to be simply the general definition applied to the poset $(\mathcal{P}(W), \subseteq)$. The difference is in the nontriviality conditions we impose on filters and ideals here, which are stronger than in the general theory of partially ordered sets.

Definition 1.2. A filter $\mathcal{F}$ is an *ultrafilter* if, for all $A \subseteq W$, either A is in $\mathcal{F}$ or the relative complement $A^c = W \setminus A$ is in $\mathcal{F}$.

Proposition 1.3. A filter is an ultrafilter if and only if it is a maximal filter, that is, a filter that is not properly contained in any larger filter.

Ultrafilters have a partition property that does not hold for all filters, in general.

Proposition 1.4. If $\mathcal{F}$ is an ultrafilter, $A \in \mathcal{F}$, and $A = A_1 \cup A_2 \cup \cdots \cup A_n$, then $A_i \in \mathcal{F}$ for some $i \leq n$.

We wish to think of $\mathcal{F}$ as measuring "largeness": the sets in $\mathcal{F}$ are called "$\mathcal{F}$-large". With this interpretation, the axioms for a filter say that:

1. $\emptyset$ is not $\mathcal{F}$-large and W itself is $\mathcal{F}$-large.
2. The intersection of two $\mathcal{F}$-large sets is itself $\mathcal{F}$-large.
3. If a set has an $\mathcal{F}$-large subset, then the set is $\mathcal{F}$-large.

Proposition 1.5. Given a filter $\mathcal{F}$, let $\mathcal{I}_{\mathcal{F}} = \{A \subseteq W : A^c \in \mathcal{F}\}$. Then $\mathcal{I}_{\mathcal{F}}$ is an ideal, called the *dual ideal* of $\mathcal{F}$.

Each ideal $\mathcal{I}$ gives a sense of "smallness": sets in $\mathcal{I}$ are "$\mathcal{I}$-small". In general, a set $A \in \mathcal{P}(W)$ is $\mathcal{F}$-large if and only if A^c is $\mathcal{I}_{\mathcal{F}}$-small.

Proposition 1.6. If $\mathcal{F}$ is an ultrafilter, and $\mathcal{I}_{\mathcal{F}}$ its dual ideal, then $\mathcal{I}_{\mathcal{F}}$ is a maximal ideal, and every subset of W is either $\mathcal{F}$-large or $\mathcal{I}_{\mathcal{F}}$-small, but not both.

If $\mathcal{F}$ is not maximal, there will be some sets that are neither $\mathcal{F}$-large nor $\mathcal{I}_{\mathcal{F}}$-small. Some of these sets are of particular interest.

Definition 1.7. A set $A \in \mathcal{P}(W)$ is called $\mathcal{F}$-*stationary*, or just *stationary*, when $A \cap B \neq \emptyset$ for every $B \in \mathcal{F}$.

Proposition 1.8. If $\mathcal{F}$ is a filter, an arbitrary subset of W is $\mathcal{F}$-stationary if and only if it is not $\mathcal{I}_{\mathcal{F}}$-small.

Proposition 1.9. A filter $\mathcal{F}$ is an ultrafilter if and only if every $\mathcal{F}$-stationary set is $\mathcal{F}$-large.

2 Examples

In this section we give four examples that together demononstrate how the definitions in the previous section apply in different areas of mathematics.

Example 2.1. The *Fréchet filter* $\mathcal{F}^\infty$ is the collection of cofinite subsets of $\mathbb{N}$. A set is $\mathcal{I}_{\mathcal{F}^\infty}$-small if and only if it is finite. A set is $\mathcal{F}^\infty$-stationary if and only if it is infinite.

Example 2.2. Let $\mathcal{M}$ be the collection of all subsets of $[0, 1]$ that have Lebesgue measure 1. Then $\mathcal{M}$ is a filter. A set is $\mathcal{I}_{\mathcal{M}}$-small if and only if it has measure 0. A set is $\mathcal{M}$-stationary if and only if it has positive measure.

Neither the Fréchet filter nor $\mathcal{M}$ is an ultrafilter. For $\mathcal{F}^\infty$, the set of even numbers is stationary but not large. For $\mathcal{M}$, the interval $(0, 1/4)$ is stationary but not large.

Example 2.3. This example is from set theory. Given an uncountable ordinal κ, and a set $A \subseteq \kappa$, we say that:

- A is *unbounded* in κ if $\sup A = \kappa$

- A is *closed* in κ if $\sup B \in A$ whenever B is a nonempty subset of A with $\sup B < \kappa$.

A nonempty set $A \subseteq \kappa$ is a *club set* if it is closed and unbounded in κ. The collection of all subsets of κ that have a club subset is a filter known as the *club filter*. A set is stationary in this filter if and only if it has nonempty intersection with every club subset of κ. The term "stationary" is most commonly encountered in this setting. ZFC proves that the club filter on an uncountable cardinal is not an ultrafilter.[1]

Example 2.4. If G is a nonempty subset of W then $\mathcal{F}_G = \{A \subseteq W : G \subseteq A\}$ is a filter on W. Filters of this sort are called *principal*. The filter $\mathcal{F}_G$ is an ultrafilter if and only if $|G| = 1$. A set is $\mathcal{F}_G$-large if and only if it contains G as a subset. A set is $\mathcal{F}_G$-stationary if and only if it has nonempty intersection with G.

3 Filter quantifiers

It is often the case that we want to know whether the set of elements with a given property is, or is not, an element of a some filter.

Example 3.1. Suppose that (a_n) is a sequence of real numbers and a is a real number. The standard definition says that (a_n) converges to a if and only if for every open interval I containing a, the sequence (a_n) is eventually in I. This can be seen equivalent to the claim that for every open interval I containing a, the set $P_I = \{n : a_n \in I\}$ is $\mathcal{F}^\infty$-large, where $\mathcal{F}^\infty$ is the Fréchet filter from Example 2.1. Similarly, a is a cluster point of (a_n) if and only if, for every open interval I containing a, the set P_I is $\mathcal{F}^\infty$-stationary.

To define filter quantifiers, we will need a sort of formal language.

Definition 3.2. An *atomic proposition* $P(x_1, x_2, \ldots, x_n)$ is simply a subset P of W^n. For each tuple $(x_1, \ldots, x_n) \in W^n$, we say that $P(x_1, \ldots, x_n)$ *holds* if $(x_1, \ldots, x_n) \in P$. The set of *propositions* is the smallest set such that:

1. Every atomic proposition is a proposition.
2. If P is a proposition then so is $\neg P$, which holds if and only if P does not hold.
3. If $P(x_1, \ldots, x_n)$ and $Q(x_1, \ldots, x_n)$ are propositions, then $P \cap Q$ is a proposition that holds, for fixed $(x_1, \ldots, x_n)$, if and only if both P and Q hold.
4. If $P(x_1, \ldots, x_n)$ and $Q(x_1, \ldots, x_n)$ are propositions, then $P \cup Q$ is a proposition that holds, for fixed $(x_1, \ldots, x_n)$, if and only if at least one of P or Q holds.

[1] An short proof is given by Andres Caicedo at *http://math.stackexchange.com/a/15305/630* .

5. If $P(x_1, \ldots, x_n)$ is a proposition then

$$Q(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = (\forall_{\mathcal{F}}\, x_i)P(x_1, \ldots, x_n)$$

   and

$$Q(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = (\exists_{\mathcal{F}}\, x_i)P(x_1, \ldots, x_n)$$

   are propositions, for $i \leq n$.

Definition 3.3. The semantics of the filter quantifiers $\forall_{\mathcal{F}}$ and $\exists_{\mathcal{F}}$. are given by these rules:

- $(\forall_{\mathcal{F}}\, x_i)P(x_1, \ldots, x_n)$ holds, for fixed $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$, if and only if the set $\{x_i : P(x_1, \ldots, x_n) \text{ holds}\}$ is $\mathcal{F}$-large.
- $(\exists_{\mathcal{F}}\, x_i)P(x_1, \ldots, x_n)$ holds if and only if $(\forall_{\mathcal{F}}\, x_i)\, \neg P(x_1, \ldots, x_n)$ does not hold.

We can give an alternate characterization of $(\exists_{\mathcal{F}}\, x)$.

Proposition 3.4. If $\mathcal{F}$ is a filter, then $(\exists_{\mathcal{F}}\, x)P(x)$ holds if and only if P is $\mathcal{F}$-stationary.

Example 3.5. Continuing Example 3.1, we have that a sequence (a_n) of real numbers converges to a real a if and only if $(\forall_{\mathcal{F}^\infty}\, x)P_I(x)$ holds for every open interval I containing a, and a is a cluster point of (a_n) if and only if $(\exists_{\mathcal{F}^\infty}\, x)P_I(x)$ holds for every open interval I containing a.

In the special case of $\mathcal{F}^\infty$, $(\forall_{\mathcal{F}^\infty}\, x)$ is usually written as $(\forall^\infty x)$ (pronounced "for all but finitely many x") and $(\exists_{\mathcal{F}^\infty}\, x)$ is usually written as $(\exists^\infty\, x)$ ("for infinitely many x").

Example 3.6. Let W be a nonempty set, let $\mathcal{W}$ be the principal filter $\{W\}$, and let $P \subseteq W$. Then the usual quantification

$$(\forall x \in W)P(x)$$

that says all elements of W are in P is equivalent to $(\forall_{\mathcal{W}}\, x)P(x)$. The usual existential quantification

$$(\exists x \in W)P(x)$$

that says at least one element of W satisfies P is equivalent to $(\exists_{\mathcal{W}}\, x)P(x)$.

The filter $\mathcal{W}$ is the smallest filter on W, and similarly the ordinary quantifiers provide an extreme example of filter quantifiers. In general, a filter quantifier over a larger filter $\mathcal{F}$ makes it easier for $(\forall_{\mathcal{F}}\, x)P(x)$ to hold, because we only need a set of witnesses that is in $\mathcal{F}$, rather than needing all of W. But, in general, $\mathcal{F}$ makes it more difficult for $(\exists_{\mathcal{F}}\, x)$ to hold, because we need an $\mathcal{F}$-stationary set of witnesses, not just one.

The next proposition gives several additional ways in which filter quantifiers resemble ordinary quantifiers, and in which quantifiers over ultrafilters have even stronger properties.

Proposition 3.7. Let $\mathcal{F}$ be a filter on W and let $P(x)$ and $Q(x)$ be propositions with a distinguished variable x and the same number of additional variables that are not shown.

1. *Duality:* $(\forall_{\mathcal{F}}\, x)P(x)$ holds if and only if $(\exists_{\mathcal{F}}\, x)\neg P(x)$ does not hold. Thus filter quantifiers have the same duality relations as ordinary quantifiers:
$$(\forall_{\mathcal{F}}\, x)P(x) \equiv \neg(\exists_{\mathcal{F}}\, x)\neg P(x),$$
$$(\exists_{\mathcal{F}}\, x)P(x) \equiv \neg(\forall_{\mathcal{F}}\, x)\neg P(x).$$

2. *Weakening:* the following implications hold:
$$(\forall\, x)P(x) \implies (\forall_{\mathcal{F}}\, x)P(x) \implies (\exists_{\mathcal{F}}\, x)P(x) \implies (\exists x)P(x).$$

3. *Conjunction:* $(\forall_{\mathcal{F}}\, x)[(P \cap Q)(x)]$ holds if and only if both $(\forall_{\mathcal{F}}\, x)P(x)$ and $(\forall_{\mathcal{F}}\, x)Q(x)$ hold.

4. *Disjunction:* If $\mathcal{F}$ is an ultrafilter, $(\forall_{\mathcal{F}}\, x)[(P \cup Q)(x)]$ holds if and only if one of $(\forall_{\mathcal{F}}\, x)P(x)$ or $(\forall_{\mathcal{F}}\, x)Q(x)$ holds.

5. *Negation:* If $\mathcal{F}$ is an ultrafilter, $(\forall_{\mathcal{F}}\, x)P(x)$ holds if and only if $(\forall_{\mathcal{F}}\, x)\neg P(x)$ does not hold.

Filter quantifiers are not similar to ordinary quantifiers in all respects. In particular they do not commute in the usual way.

Example 3.8. Consider $P = \{(x, y) \in \mathbb{N} \times \mathbb{N} : x < y\}$. Abusing notation slightly by not writing P explicitly, the proposition
$$(\forall_{\mathcal{F}^\infty}\, x)(\forall_{\mathcal{F}^\infty}\, y)[x < y]$$
holds: for every x there are infinitely many y greater than x. If we reverse the quantifiers, the new proposition
$$(\forall_{\mathcal{F}^\infty}\, y)(\forall_{\mathcal{F}^\infty}\, x)[x < y]$$
does not hold: no y is greater than infinitely many x. In fact, even the weaker proposition $(\exists y)(\exists_{\mathcal{F}^\infty}\, x)[x < y]$ does not hold.

4 Applications

Beyond giving a useful general perspective on concepts such as convergence (Example 3.5), filter quantifiers can also make definitions more accessible by using a more informative notation.

Example 4.1. Let $\beta\mathbb{N}$ be the set of all ultrafilters on $\mathbb{N}$. This set $\beta\mathbb{N}$, with an appropriate topology, is the Stone–Čech compactification of $\mathbb{N}$, and arises in many areas of mathematics.

In infinitary combinatorics, a operation $\oplus$ is defined on $\beta\mathbb{N}$, followed by a construction of ultrafilters $\mathcal{U}$ for which $\mathcal{U} \oplus \mathcal{U} = \mathcal{U}$. These *idempotent ultrafilters* can then be used to prove combinatorial results such as Hindman's theorem.[2]

The definition of $\oplus$, stated in set builder notation, is somewhat opaque:

$$\mathcal{U} \oplus \mathcal{V} = \{A \subseteq \mathbb{N} : \{x : \{y : x + y \in A\} \in \mathcal{V}\} \in \mathcal{U}\}.$$

This definition becomes much more direct with filter quantifiers:

$$\mathcal{U} \oplus \mathcal{V} = \{A \subseteq \mathbb{N} : (\forall_{\mathcal{U}}\, x)(\forall_{\mathcal{V}}\, y)[x + y \in A]\}.$$

Example 4.2. In model theory, *Łoś's theorem* is the key tool for studying ultraproducts. Suppose that $\{M_i : i \in I\}$ is an indexed family of models of a first-order theory. Let $\mathcal{U}$ be an ultrafilter on the index set I. The *ultraproduct* is the structure

$$\prod_{i \in I} M_i \,/\, \mathcal{U}$$

which is obtained by treating two elements of the Cartesian product $\prod_{i \in I} M_i$ as equivalent if they are equal on a set of indices in $\mathcal{U}$:

$$(a_i : i \in I) \equiv_{\mathcal{U}} (b_i : i \in I) \quad \iff \quad (\forall_U\, i)[a_i = b_i].$$

Łoś's theorem states that a sentence ϕ is true in $\prod_{i \in I} M_i \,/\mathcal{U}$ if and only if the set of $i \in I$ for which M_i satisfies ϕ is in $\mathcal{U}$:

$$\left(\prod_{i \in I} M_i \,/\, \mathcal{U}\right) \vDash \phi \quad \iff \quad (\forall_{\mathcal{U}}\, i)[M_i \vDash \phi].$$

[2] For a proof using ultrafilters and filter quantifier notation, see Imre Leader, *Ramsey Theory*, 2013, pp. 19–23, lecture notes transcribed by Stiofáin Fordham, at *http://www.maths.tcd.ie/~stevef/ramsey.pdf*.